Covering a chessboard with dominoes
Eight problems

Problem 1
Is it possible to cover a whole chessboard with dominoes?

Problem 2
One corner has been removed from a chessboard. Is it possible to cover the remaining portion of the board with dominoes so that each domino covers exactly two squares? What if two opposite corners are removed?

Problem 3
Two arbitrary but adjacent squares have been removed from a chessboard. Is it possible to cover the remaining portion of the board with dominoes?

Problem 4
Two arbitrary squares of different colors have been removed from a chessboard. Is it possible to cover the remaining portion of the board with dominoes?

Problem 5
Two squares of each color have been removed from a chessboard. Is it still possible to cover the remaining portion of the board with dominoes?

Problem 6
Three squares of each color have been removed from a chessboard, so that the chessboard does not split into two or more separate pieces. Is it still possible to cover the remaining portion of the board with dominoes?

Problem 7
A domino has two edges, a long edge and a short edge. Two adjacent dominoes must be in one of the following three configurations: long edge to long edge, short edge to short edge and long edge to short edge. When tiling a chessboard with dominoes, what is the minimum number of long-edge to long-edge pairs?

Problem 8 (bonus, very difficult)
Prove that in any cover of a whole chessboard with dominoes, the number of horizontal dominoes with a black left square equals the number of horizontal dominoes with a white left square.

Covering a chessboard with dominoes
Solutions

Problem 1
Is it possible to cover a whole chessboard with dominoes?

A chessboard consists of 64 squares - eight lines of eight squares. We can cover the board with 32 dominoes by putting four horizontal dominoes on each line.

Problem 2
One corner has been removed from a chessboard. Is it possible to cover the remaining portion of the board with dominoes so that each domino covers exactly two squares? What if two opposite corners are removed?

If a corner is removed from the chessboard, then the number of squares is odd. Since a domino consists of two squares, it will be impossible to cover the amputated board with dominoes so that each domino covers exactly two squares: one would necessarily have one half outside the board.
If two opposite corners are removed, the amputated board consists of 62 squares, so the problem stated above doesn't arise. Now, paint the squares black and white as on a regular chessboard. The opposite corners are the same color, as a consequence the amputated board will consist of 30 black squares and 32 white squares or vice versa. But a domino will always cover a black square and a white square, so it is impossible to cover the amputated board with dominoes.

Problem 3
Two arbitrary but adjacent squares have been removed from a chessboard. Is it possible to cover the remaining portion of the board with dominoes so that each domino piece covers exactly two squares?

By studying different positions for the removed squares, it's easy to get convinced that it is still possible. But this is not a satisfactory way to prove it. If we consider the basic tiling of the chessboard one gets with 32 horizontal dominoes, three situations arise when removing two arbitrary but adjacent squares.

1. If the two removed squares are under the same tiling domino, then the amputated board is obviously tileable with dominoes.

2. If the two removed squares are on the same row but under two horizontally adjacent dominoes, then we just have to place one horizontal domino on the row below or above those two squares (depending on the parity of the number of the row), one vertical domino to the right and one to the left. We get the configuration shown on the right, or its horizontal mirror image, which fills a 4×2 part of the board, the rest of the horizontal dominoes remaining in the same position.

3. If the two removed squares are on the same column, under two vertically adjacent dominoes, then we just need to place a vertical domino over the two other halves of those dominoes.

Problem 4

Two arbitrary squares of different colors have been removed from a chessboard.
Is it possible to cover the remaining portion of the board with dominoes so that each domino piece covers exactly two squares?

It is always possible. You can see it by drawing two forks as shown on the diagram. This splits the chessboard into a chain of alternating squares. The chain can be traversed by starting from any square and covering two squares at a time with a piece of domino. The two removed squares must be a different color, so there will always be an even number of squares along the chain between them. Start from one side of one of the removed squares and place dominoes along the chain until you get to the other one. Do the same from the other side. Note that this would not be possible if the two removed squares were the same color, as the chains between them would have an odd number of squares.
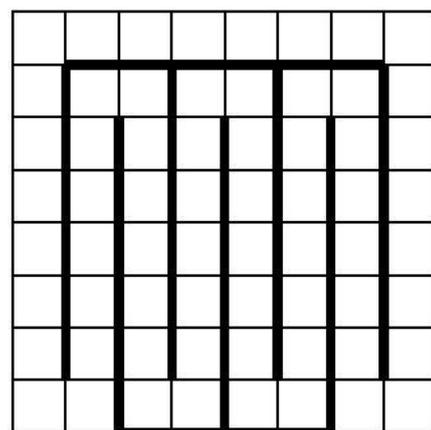
Problem 5

Two squares of each color have been removed from a chessboard. Is it still possible to cover the remaining portion of the board with dominoes so that each domino piece covers exactly two squares?

The problem has two different answers if we allow the removed squares to isolate one part of the board or not. If we allow this, then the tiling is not always possible. Consider for example the situation were one corner is surrounded by two removed squares of the same color, the two other ones being anywhere else on the board. Then, the isolated corner is obviously not tileable with dominoes and so the amputated board is not.
If we restrict the situation to the cases where the board doesn't fall apart, it seems that the tiling is always possible, but the proof is not easy to write.

Problem 6

Three squares of each color have been removed from a chessboard so that the chessboard does not split into two or more separate pieces.
Is it still possible to cover the remaining portion of the board with dominoes so that each domino piece covers exactly two squares?

As we will show with one special configuration this is not always possible to achieve.
The figure on the right represents the lower right corner of the board. Remove the three black squares marked with a
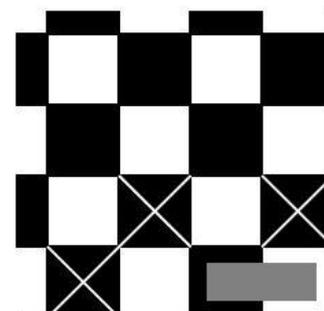
right cross, and the three white squares anywhere else on the board so that it does not split into two or more pieces. The only way to cover the white square on the lower right angle is to put a domino as shown with a crosshatched rectangle. It will then be impossible to cover the white square to its left.

Problem 7

A domino has two edges, a long edge and a short edge. Two adjacent dominoes must be in one of the only three possible configurations: long edge to long edge, short edge to short edge and long edge to short edge. In a domino tiling of a chessboard, what is the minimum number of long-edge to long-edge pairs?
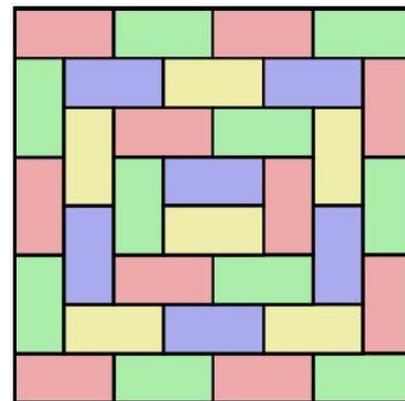
The tiling shown on the right has only one long edge to long-edge pair. This is the smallest possible number of such pairs since it is impossible to do a tiling with none. Indeed, if you start with a domino in one corner, avoiding long-edge to long-edge pairs you will have to put all the dominoes as shown on the figure, except for the four middle squares. Then we need to place two dominoes on these squares, which will give us three (four?) long-edge to long-edge pairs in one case and one in the other.

Problem 8

Prove that for any pattern of covering a chessboard with dominoes, the number of horizontal dominoes with a black left square equals the number of horizontal dominoes with a white left square.

The proof of this property is given by E. W. Dijsktra:

For a given partitioning, consider one of the seven lines that separate two adjacent columns; it divides the board into a left and a right part. We observe that:

- since the column length is even, each column, and hence the left part, contains equal numbers of black and white squares ;
- since a domino consists of a black and a white square, the dominoes to the left of the line comprise equal numbers of black and white squares.

Combining the two observations, we conclude that, among the horizontal dominoes crossed by the line, there are as many with a black left square as with a white one. Since each horizontal domino is crossed by exactly one of the seven lines, we conclude by summation there are as many dominoes with a black left square as dominoes with a white left square.