The solar system[1]

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[1] For more details and developments, in French, don’t hesitate to consult the following website http://www.discip.ac-caen.fr/phch/culture/syst_solaire/CONFSYSO.htm

A/ FROM ARISTOTLE TO ISAAC NEWTON

We have to be fast and step over the centuries.

- In ancient Greece, different ideas are at odds with each other: in disagreement with ARISTARQUES OF SAMOS, **ARISTOTLE** thought the Earth was motionless.

His idea of force is not very clear; besides, he confuses speed and variation of speed, i.e. acceleration. Means of transport are horse-riding and sailing: he notices the centrifugal force and points out that, if a boat accelerates because of a sudden squall of wind, the sailor tends to fall backwards.

This will be the main argument for the absolute motion of our planet: if it moved, everything on it would stay behind.

ARISTOTLE is short sighted because the world he observes is small, limited to the Peloponnese and Macedonia. He hasn’t the faintest idea of astronomical distances. The Earth seems infinite and is the only motionless body in the Universe. That’s why stillness is nearly sacred, absolute and opposed to motion. In other words, motion and motionless are special properties of bodies, like temperature: something can be still or in motion, hot or cold.

Let us add that, for ARISTOTLE, stars float in the sky: if they don’t fall, it’s because they are light.

This geocentricism will please the Roman Catholic Church and will last for a very long time.

- The Greek astronomer **PTOLEMY** tries to explain geometrically the motions of the nearest planets and the Dane **TYCHO-BRAHE** makes observations that will remain the best for a long time. Neither the former nor the latter will doubt ARISTOTLE’s theory. TYCHO-BRAHE’s works are important because his measures are so precise that one begins to have ideas about astronomical distances. TYCHO notes that comets pass behind the Moon and in a backward direction: therefore, the crystal, rigid, concentric, celestial spheres on which stars were thought to be painted do not exist.
- **COPERNICUS** clearly proposes an heliocentric solar system.
- **GALILEO** builds the foundations of modern physics. First, he improves his visual equipment enough to see thousand of celestial bodies and especially to notice that the Moon has mountains, Jupiter (like the Earth) has satellites and the Sun has spots. GALILEO finally understands that the would-be perfection of the Universe is a lie, that there are bodies in the sky like the Earth which therefore no longer has any need to be the centre of Universe and the only motionless body. So he adopts COPERNICUS’ theory but proceeds much further. Since the Earth is no longer considered as motionless, the ostensible stillness of bodies is just a shared motion. Motion and motionless are no longer absolute properties.

GALILEO deserves the huge credit of feeling that experimental checking is necessary and his famous experiment of the stone dropped from the top of a mast of a boat, berthed or sailing, allows him to refute ARISTOTLE’s argument.

- At the same time, using TYCHO’s measurements, **KEPLER** discovers, after unbelievable calculation, the 3 laws that bear his name; they are descriptive laws, i.e. they are not based on physical principles:
  - ∗ each planet moves around the Sun in an ellipse with the Sun at one focus;
  - ∗ the radius vector from the Sun to the planet sweeps out equal areas in equal intervals of time;
  - ∗ the squares of the periods of any two planets are proportional to the cubes of the semi-major axes of their respective orbits.
- **NEWTON** uses KEPLER’ results and GALILEO’s principles to write down the laws of dynamics. These laws and gravitation theory make possible the understanding of planet~~s~~ motions in the solar system (periodic, independent of the mass of the planets) in fixed ellipses called keplerian orbits. The properties of the Earth-Moon (and more generally of the 2 body systems) are roughly interpreted.

NEWTON doesn’t proceed further. He wonders about the solar system stability which seems strongly doubtful because of the hugeness of gravitation forces. He even supposes that an almighty intelligence has to intervene from time to time to put things in order. He probably lacks time. Besides, the analysis he discovered with LEIBNIZ has still to be improved.

- Irregularities are pointed out early (HALLEY) on Saturn and Jupiter's revolution periods. But, it is not easy to raise questions about NEWTON’s work. Besides, it is not obvious that discrepancy does not have its roots in the law of conservation. In the XVIIIth century, even if NEWTON becomes an object of worship, gravitation theory has not yet gained the universality it has now. It was not totally a heresy to think that discrepancy could be interpreted with a law in $1/r^{2+\varepsilon}$ with $\varepsilon << 2$.

Changing COPERNICUS’s circular orbits to KEPLER’s ellipses means perfection has to be abandoned. GALILEO was puzzled about that. It happens again and we have to accept unclearly defined curves and non periodic motions.

B/ PIERRE-SIMON DE LAPLACE: THE STABILITY OF THE SOLAR SYSTEM

1° A stupendous century

Just after NEWTON, a few basic discoveries are made. But it soon turns out that the number and the quality of scientists will be outstanding in that end of the XVIIIth century. CLAIRAUT, EULER, d’ALEMBERT, the BERNOULLIs and LAGRANGE are **LAPLACE**’s predecessors. And his fellow workers or pupils are famous: BIOT, GAY-LUSSAC, MALUS, FOURIER, ... If we bear in mind that the scientists LAPLACE meets are MONGE, LAVOISIER, BERTHOLLET, ..., we realise the environment is highly favourable; and we note that LAPLACE knew how to surround himself with excellent scientists.

2° A difficult problem: LAPLACE’s purpose is to explain:

- a slight but perfectly detected discrepancy in planet motions. Besides, a few astronomers pretend they noticed a “secular variation” (i.e. monotonous, in the sense of functional analysis) of orbital parameters of Saturn and Jupiter and a few scientists - as famous as EULER, d’ALEMBERT and LAGRANGE - claim they have proved this variation; but their results are poorly consistent and LAPLACE who does their calculation again finds mistakes.
- the solar system stability, i.e. to establish if the different interactions between the bodies only shift them slightly and temporarily on their mean orbits or if these changes are radical and irreversible.

LAPLACE perfectly knows the laws of mechanics. His writings are clear: the concepts of energy and angular momentum are perfectly understood. He may sometimes be difficult for a modern reader; but it’s often only because of his outdated vocabulary: for example, linear momentum is called force and force, in the modern sense, accelerating force.

As a rough approximation, let us assume the Sun and planets to be points, the Sun to be in O centre of co-ordinates with an infinite mass. Planet number i is in G_i ($\overrightarrow{OG}_i = \vec{r}_i$) and its mass is m_i. If K is global kinetic energy, total mechanic energy (hamiltonian) is:

$$H = K - \sum_i G\frac{m_i M}{r_i} - \frac{1}{2}\sum_i\sum_j G.\frac{m_i . m_j}{\left|\vec{r}_i - \vec{r}_j\right|} .$$

The motion of G_i is given by NEWTON's third law:

$$m_i \vec{a}_i = -G\frac{m_i M \vec{r}_i}{\left|\vec{r}_i\right|^3} - \sum_j G\frac{m_i . m_j(\vec{r}_i - \vec{r}_j)}{\left|\vec{r}_i - \vec{r}_j\right|^3},$$

which can be written (x_i, y_i z_i components of $\vec{r}_i$):

$$m_i \ddot{x}_i = -G\frac{m_i M x_i}{\left(x_i^2 + y_i^2 + z_i^2\right)^{3/2}} - \sum_j G\frac{m_i m_j (x_i - x_j)}{\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{3/2}}$$

idem y and z.

For the so-called two body problem (the Sun + a planet or the Earth + the Moon) solved by NEWTON, we have 2 laws of conservation: energy (without taking friction into account) and angular momentum.

A model as simple as the 3 body problem cannot be solved in the common sense of the term.

So the solar system is not simple at all. The difficulties are numerous:

- the number of interacting bodies is both big and unclearly defined. Even if asteroids are unknown and all the planets not yet discovered, how many are involved?
- initial conditions are unclear.

However, the problem is not hopeless:

- the system is frictionless to a very good approximation. Internal friction does exist and tides can be regarded as the basic phenomenon which explain the synchronisation between self-rotation and motion on orbits (the Moon always shows us the same face and the couple Pluto-Charon, amazingly enough, does the same).
- the laws of gravitational interactions are well known and at least formal equations can be written. Approximate solutions have been established by KEPLER and NEWTON: they are 0 order approximation.
- the numerical values of a few parameters remain small: eccentricities of the orbits, planets masses (compared to the Sun’s) as well as the energies of the interactions between the planets (compared with the one between them and the Sun). We stay close to the keplerian system, i.e. to an integrable system.

Not taking into account the interactions between the planets, we can reduce the equations: planets masses do vanish. It’s the 0 order approximation. LAPLACE's theory is order one: in the equation of planet number i, mass m_i can be reduced but it appears in the equation for planet $j \neq i$. Thus planet n° i motion depends on its own mass m_i, at an order at least equal to 1.

To prove the stability of the solar system, LAPLACE doesn't have many solutions: either he slightly shifts a planet from its position and shows the evidence of a back pulling force or, which is the same, the evidence of a minimum of potential energy, for instance looking parabolic. LAPLACE can’t use more sophisticated criteria; neither can he be helped by calculators. He expands the energy in a power series of small parameters or of the shift with the values they would have in keplerian approximation. In fact, LAPLACE chooses local co-ordinates that follow a system of coupled linear differential equations. Solving them needs a change of co-ordinates found by diagonalizing a 7×7 matrix (Neptune has not yet been discovered); it’s the same calculation for the modes for coupled oscillators. A mathematics theorem specifies the eigenvalues of the system are purely imaginary, which means the modes are sinusoidal: their period happens to be between 50 000 years and a few million years. The physical co-ordinates are not sinusoidal but their mean value is nil.

At the same time he proves similarly likely stability for the solar system, (no need of an almighty intelligence), LAPLACE solves the problem of the secular variation of planetary orbits parameters. He also proves this result is not totally inconsistent with the measures of the astronomers who thought they were pointing out a monotonous change in the planet~~s~~ motions.

Planet~~s~~ motions are in trajectories which remain near keplerian orbits: they are practically ellipses whose axes turn and parameters change slowly. They are not closed curves.

3° From LAPLACE to WEIERSTRASS and POINCARÉ

•**Time boundaries.** LAPLACE perfectly understands that the limited expanding of the energy in a power series of "small parameters" (i. e. ratio $\varepsilon_i = m_i / M$ and orbits eccentricities) is a rough approximation. He underestimates the underlying difficulties but he can hardly do differently. In fact, reading LAPLACE, we can sometimes think he uses power series of anything. But his cleverness and intuition make him succeed.

The solutions based on the cutting off of the series may give a good approximation for a finite time but don't prove everlasting stability. LAPLACE's results are reasonably good for times of about a few hundred thousand years. As one hasn't the faintest idea about the age of the Universe, it's understandable that many people are satisfied with this result. LAPLACE's successors (LE VERRIER, NEWCOMB, LINSTEDT and HILL) try to proceed further with the power series to a bigger level.

In fact, that doesn't give a better approximation: the result, far from being improved, gets worse.

•**Convergence problems** don't worry mathematicians before CAUCHY. LAPLACE's calculations were limited to the first terms. It could be hoped that expanding in a power series until a high rank would give an arbitrarily good precision. But LE VERRIER was one of the first to notice that the LINSTEDT series (since they bear that name) do not generally converge. In other words, there are terms whose denominator is very small and who make the convergence impossible.

That can be seen as resonance phenomena, well known in physics and here very sharp because friction is very small. In celestial mechanics, the problem is complex because these resonances can as well prove instabilities with the kicking out of a planet as the locking on an orbit. Jupiter plays the main role. The American astronomer Daniel KIRKWOOD will soon notice that there are “holes” in the asteroid belt. (KIRKWOOD gaps). These holes fit with bodies whose periods around the Sun would be in the ratios 3/1, 7/3and 9/4 with Jupiter's. It seems reasonable to think that Jupiter's repeated actions move away the bodies far from the resonance orbits. However other resonances fit with a local excess of asteroids. And there is the Jupiter- Saturn couple we will soon talk about again.

In the XIXth Century, the solar system and the solving of the motion equations of the planets worry many scientists, especially mathematicians: DIRICHLET tells his student KRONECKER he has found a new method of solving those equations and has proved the general stability of the solar system. But he dies without leaving anything. KRONECKER and WEIERSTRASS try to solve the problem without success. A prize (WEIERSTRASS and MITTAG-LEFFLER) is promised by King OSCAR II of Sweden in 1890 to anyone who would give a significant improvement on the topic: *for any system of massive points interacting according to NEWTON's laws, assuming there is no collision, give in function of time the co-ordinates of the points in the form of uniformly converging power series whose terms are written with known functions*.

DIRICHLET so renowned that WEIERSTRASS doesn't doubt he solved the problem *not with long and complicated calculations but by expanding a simple and basic idea we can reasonably hope to find it again thanks to a persevering and keen search*.

C/ RELAXATION: SIMULATED TRAJECTOIRES

Expecting H. POINCARÉ, we can try to achieve a simulation for we do have calculators which he did not. Using a common tool like EXCEL or WORKS, we can get an idea of the difficulty for three bodies.

We can begin with simple mathematically defined curves: circles, ellipses and unperfect ellipses whose characteristics (eccentricities, major axis) have a slow evolution.

We'll get such paths later. Now, let us try with physics. We have the mechanics equations, i.e. NEWTON's third law.

I try with MATLAB, an impressive scientific tool. After a day spent with someone used to it, I didn't get much more than with EXCEL though we can get simultaneously the paths of all the planets involved. The algorithm used is likely a RUNGE-KUTTA type of 4 order on MATLAB. The little DIY on EXCEL-WORKS is a lowest order RUNGE-KUTTA.

I first try the algorithm with one planet and give the initial conditions (velocity and position) of the Earth. We do get an ellipse with a very small eccentricity, i.e. nearly a circle (mind the scales on x- and -axes).

Now I try the so-called LAGRANGE problem: fixed attracting Suns and a moving planet thrown with chosen conditions. This simple example clearly emphasises that a weak variation of one component of the initial velocity may involve a very big change in the shape of the paths. Finally, I try with two Jupiters in their motions round the Sun.

D/ FROM HENRI POINCARÉ TO JACQUES LASKAR

1° POINCARÉ’s works

- **Towards chaos.** POINCARÉ tries a very simplified version of the three body system: $m_1 > m_2 >> m_3$ infinitely small, i.e. without any influence on the two others. The problem is still very hard. First, POINCARÉ thinks he found a sort of stability, but his early paper contains mistakes whose corrections make him foresee an awfully complex dynamic situation which leads him to the study of chaos.
- **Initial conditions sensitivity (ICS).** Classical mechanics is often understood as the field of continuum: it's generally admitted that weak magnitude phenomena only involve a weak change in the evolution of the system. It's not always true: for instance, in pools, a slight change in the position of the muzzle velocities changes the angle of incidence and involves an important change in the motions after a few collisions. (the Russian mathematician SINAI gave his name to this sort of billiards). In fact, the models are elementary (very often linear) and that is the reason why we get these results. To take an

often used example, the parabolic type potential energy for the harmonic oscillator or for gravitation interaction of two bodies has a quadratic shape which does minimise the shifts.

It's no longer true for N- body (N> 2) problems. POINCARÉ did separate what classical mechanics is involved in and the problems due to the model~~s~~ types. Science philosophers sometimes say in a pedantic way that he separates classical mechanic from the linear paradigm. This ICS alters the way we consider problems, especially, but not only, the planets motions. We cannot doubt there will always be a limit in our control of the physical systems in early state or during their evolution: precision of measures, of course, but also unavoidable tiny collisions, thermalisation, ...

The ICS will (for POINCARÉ had only his pencil) show in a different light the consequences of the precision of measurement and the rounded-off numbers in the computers; these approximations are considered as unimportant: is it really the case? A simple and instructive example is the simulation of a perfect gas in the left part of a box. The right part is empty and separated from the other one by an air-proof removable wall. We open at t = 0. The molecules are discs of the same radius which hit each other or hit the walls in elastic collisions whose laws are perfectly known. At the time t_1, the whole box is full of gas. But if we reverse time and all the velocities, at $t = - t_1$, the gas has not returned into the left part of the box: the unavoidable approximations give an ostensible irreversibility.

We somehow get round the difficulty by checking, such as backwards calculations and optimisation of precision; the problem is still a real puzzle.

- **The phase space:** leaving mathematical analysis, POINCARÉ considers the phase space. In the case of the motion of a point on an x-axis, the phase space is 2-dimensional; for a system of N points and moving in physical space, the phase space 6N-dimensional: $x^{(1)}, y^{(1)}, z^{(1)}, x^{(2)}, y^{(2)}, z^{(2)}, ..., x^{(N)}, y^{(N)}, z^{(N)}, p_x^{(1)}, p_y^{(1)}, p_z^{(1)}, p_x^{(2)}, p_y^{(2)}, p_z^{(2)}, \ldots, p_x^{(N)}, p_y^{(N)}, p_z^{(N)}$.

We can consider other physical quantities in a still more complex phase space, if we take self rotation of planets in to account.

Using that phase space, POINCARÉ gives a geometrical interpretation of phenomena. He emphasises that the two body problem is solvable because of the conservation of two physical quantities: energy E and angular momentum $\vec{L}$. In the phase space, the path, which has two constraints whose mathematical expression give two surfaces, is clearly defined: it's the intersection of the two surfaces.

The number of bodies in interaction increases, POINCARÉ demonstrates that the number of conservation laws is still the same: 2 which give 4 relations (E, L_x, L_y L_z). In the phase space, the trajectory has few constraints and can go where it wants to. That property with the ICS results in the paths in the space phase being regarded as stochastic, though the system is perfectly deterministic. The problem is not integrable and not only because of technical difficulties. In the simplified 3 body problem (fix Sun and 2 interacting planets) , the phase space is 12 dimensional (2x3 for position components, 2x3 for linear momenta). The path is in 12-4 = 8 dimensional subspace. This space is really abstract and POINCARÉ was difficult to follow, even for WEIERSTASS.

In the classical case of one dimensional problem, if the system is frictionless, the centre O is not attracting (no centripetal spiral) and the path is roughly an ellipse. In the general case (H non explicitly depending on time), POINCARÉ shows that the evolution with a H constant surface is a torus on which the path whirls. This trajectory cannot intersect itself because, in such a point, the same conditions must give the same evolution. POINCARÉ studies meticulously the paths in the phase space by intersecting them with planes (called POINCARÉ's planes) and turns an analysis problem (LINSTEDT's series convergence) into a geometry and topology problem. He finds how one can proceed from point P_k , intersection of the path with POINCARÉ's plane (number k passage) to point P_{k+1} (number k+1 passage). The knowledge of that transformation (POINCARÉ's functionnal) is a beginning of a solution of our problem.

One year after POINCARÉ's death, the Finnish mathematician Karl SUNDMAN gives a better answer to the test for the MITTAG-LEFFLER and WEIERSTRASS prize by finding a particular solution to the 3-body problem using a converging power series (in fact, this solution converges too slowly to be of any practical importance). The topic and POINCARÉ's works are then forgotten until George BIRKHOFF in the United States in the late 20s; but the true rediscovery is made by the soviet school of Andrei KOLMOGOROV as early as 1935.

2 ° KAM theorem and chaos

- **KOLMOGOROV (1954), ARNOLD (1963) and MOSER (1967):**

Somehow obsessed by the non-convergence of LINSTEDT's series and the non integrability of the N body (N>2) problem, POINCARÉ hastily concludes stability is impossible for the solar system. However, the main resonance is the Jupiter-Saturn couple whose "years" are in the ratio of 2 to 5. Periodically, both planets are in the same position and we'd expect the perturbations to be amplified. BIOT once predicted somewhat hurriedly that a little perturbation on Saturn's orbit would eject it and WEIERSTRASS thought and pointed out that the commensurable character of the periods can't have a great sense since the precision of measurement doesn't allow a conclusion between rationality and irrationality of the ratio. However the couple Jupiter-Saturn ~~is~~ still exists.

The approximation of irrational numbers by rational ones does play a great role. KOLMOGOROV intuitively guesses a solution in 1954. His pupil Vladimir ARNOLD in 1963 and Jürgen MOSER in the late sixties state a result known as KAM theorem or KAM tori. There can be no question of studying that works which are only understandable to experts, because this theorem which, on a few points, is the opposite of POINCARÉ's brilliant intuition, is very difficult. It roughly means that order is much stronger than what could be thought and emphasises how mechanical frictionless systems get out of stability because of perturbations:

- under special or peculiar initial conditions, series converge and motions are virtually periodic as in integrable problems;
- under other and very near conditions, instability (or chaotic areas) appears.

KAM theorem applies to any frictionless system. But no conclusion for the real solar system has ever been made. The theorem points out a maximum value for planets masses (as well as for eccentricities and angles between paths) for keeping stability: these values are very weak, much smaller than those of the planets' in the solar system. It may be objected that the boundary conditions are pessimistic but nothing proves that less strict conditions cannot practically give the same result. KAM theorem doesn't guarantee the stability of the solar system but it made it likely. But things have changed.

- **Chaos** is not apocalypse or explosion. It's just a consequence of the ICS.

What does chaos look like? Here are chaotic vibrations: it's not easy to distinguish them from random oscillations. We'll see how chaos can be detected. In the phase space, most of the times, the representative point makes several times the same loops of the path and, suddenly, the system "goes away".

The ICS involves that paths do not repeat themselves exactly where "they should". It's the case both in the simulations where approximations may have important consequences and in the true physical world where infinitesimal changes, that are too small to be controlled (collision with a particle, change of temperature, ...) may add up and make the path "escape".

In the phase space, trajectories must obey two opposite constraints: energy diminution by friction which brings them nearer and time evolution with ICS which spreads them away. That locally involves a contraction in directions in the phase space and a stretch in others. Contractions (or folding) are necessary because physical quantities are limited. Folding and stretching is a well-known transformation in topology: "baker's transformation" because it looks like what he makes to prepare his puff pastry. In the same way, there is always air (and butter) in it, the set of the paths (which do not fill the phase space in a thick way) has a strange peculiar look (strange attractor).

Another characteristic is the virtually periodic evolution with time: stretches and folding follow each other without any fundamental change. Paths are nearer and nearer, intersections with POINCARÉ's planes are more and more numerous and basically, we always see the same things. The only way of making some finite with infinite is a scale invariance: if a big line is made of one separate and of two nearer, a big optical magnification shows each one has the same structure.

- **Chaos and fractals; non-integer dimensions**

The intersections of a path with a POINCARÉ plane gather into a fractal. one of the safest means of proving the evidence of chaos is to find the dimension of that fractal: non-integer HAUSSDORFF(-BESICOVITCH) dimension (for there are other definitions). Let it be assumed that points are spread regularly in space and P one of them, centre of a circle (radius R): if they are on a curve, the number of points within the circle is proportional to R; if they are on a plane, the number is proportional to R^2. And if the number is found proportional to R^d, d is the (may be non-integer) dimension of the fractal. That makes sense only if the same phenomenon is repeated whatever the value of R: scale invariance is fundamentally bound to chaos and fractals.

- **LIAPOUNOV's exponents:** Time evolution of the system may be represented by a diagonalizable matrix L(t) whose eigenvalues are of $\exp(\lambda_i.t)$ type. For a chaotic system, the biggest real one is positive: $\lambda_{max} > 0$: that means a divergent evolution when time increases.

The determination of the exponent ($\tau = 1/\lambda_{max}$ which has a time dimension, seems to have an obvious sense) gives only calculation difficulties if a mathematical model is available: in celestial mechanics, the H operator is well known and one gets for τ a few million years (between 4 and 25).

When real experimental results (without any satisfactory mathematical model) are only available, τ calculation may be impossible. Anyway, the true sense of the exponent is difficult: τ is not always the time when a radical change may occur for a single orbit or for the whole system. However, the value of a few million years is puzzling.

However, a positive LIAPOUNOV exponent and a non-integer dimension of the fractal drawn in a POINCARÉ cut plane are good chaos signs: it was, at an early stage, suspected in the motions of small satellites (Hyperion of Saturn for instance) and of small planets (Mercury).

3° Jacques LASKAR and modern calculators

Recent progress in astronomical measurement and increasing power of computers for calculation give precise data; and more and more realistic models are used.

However, the 20 body problem (planets and satellites) is not yet solved. In all models, physicists don't try to calculate the motion of each planet but to find characteristics of these motions. Jacques LASKAR points out one would be satisfied if the time evolution of the shapes and of the positions of the orbits was known, details about the motions on these orbits being searched afterwards. Thus, he has the idea of studying a mean motion of the orbits on which he spreads uniformly the planets mass, which simplifies the equations and makes the solving possible (but with great difficulty).

It appears that for the biggest planets, the motion is steady; but for Mercury, Venus, the Earth and Mars, it's chaotic, LIAPOUNOV exponent being roughly 5 million years: the distance of two initially very close orbits is multiplied by 3 every 5 million years and a relative shift of 10^{-10} on initial conditions gets 100 % within 100 million years, which makes any forecast on such times impossible. Inversely, if the positions of a planet is to be known in 100 million years, the uncertainty on its current position has to be smaller than 10^{-40} m which is less than the smallest likely " physical " distance: PLANCK's distance $L_P = \left(\frac{\hbar.G}{c^3}\right)^{\frac{1}{2}} \approx 10^{-35}$ m .

Two mathematicians SMALE et CHIRIKOV later prove that chaos is due to Mars-Earth and Mercury-Venus-Jupiter resonances.

Things can be seen in another way. The nearly elliptic orbits change, their axes slowly turn and the paths sweep a ring-shaped area. One can say, like Ivar EKELAND, that the Earth trajectory is a ring rather than an ellipse; let us remember that we met fractals which are (only in the phase space) something between curves and surfaces.

LASKAR points out that the ring-shaped areas which contain the orbits of all the planets fit with such a perfection that there is no room for another planet: if one comes, it's ejected and that happened to the Moon.

The solar system is chaotic: that doesn't mean that all the planets will be spread away but only that any forecast is impossible beyond a certain time. Anyway, at that time, the Sun will have become a red giant and will have "swallowed" the planets.

E/ FROM (LAPLACE) NEBULA TO PLANETS

1° Nucleosynthesis: The Universe contains chiefly hydrogen. Where do the other nuclei, specially the heavy ones, come from? Nucleosynthesis is the topic of astrophysics which tries to answer that question.

- **Fusion:** A cloud of protons can contract because of gravitation if it's bigger than electrostatic repulsion; it's the case if the mass of the cloud is big enough: thus, temperature becomes high enough too.

If $T > 1{,}5 \times 10^7$ K, hydrogen fusion into helium can happen.

Fusion into bigger nuclei is more difficult: if $T > 4 \times 10^9$ K, iron can be reached. But the shape of the curve giving binding energy pro nucleon as a function of the number of nucleons shows one cannot go further.

- **Heavier nuclei synthesis** is probably due to neutron captures (${}^{A}_{Z}X + n \rightarrow {}^{A+1}_{Z}X + \gamma$): these neutrons should be produced by reactions as ^{22}Ne(α, n)^{25}Mg, ^{13}C(α, n)^{16}O, et ^{21}Ne(α, n)^{24}Mg; the best argument is the fact that " magic in neutrons " nuclei (N = 50, 82 et 126) are abundant.

Spallation phenomenon in cosmic rays i.e. collisions with very high energy particles (until ~ 10^{20} eV) can produce other nuclei.

- Such physical conditions can be can be found in the crucible which is a **supernova** .

2° Different theories: The birth of the solar system excited the imagination of many physicists. One model has proved to be the best for a few years.

- **Are the planets of the solar system pieces of a star?**

It can be imagined that a passing-by star pulls out from the Sun bits which, after getting cold, become planets (CHAMBERLAIN, MOULTON, JEFFREYS and JEANS) or, on the contrary, the Sun pulls out bits from the star. We would be direct sons of the Sun or of a star. Those models, though simple, have now been dropped:

- Calculations in hydrodynamics and thermodynamics prove that the scattering of the pulled out scraps can't give a condensation into planets: cooling is very slow and matter is quickly dispersed.
- The birth of planets from a hot body (the Sun or the star) is not compatible with the abundance of a nucleus as fragile as deuterium which doesn't exist inside the stars.

- **Formation from a rotating disc (primitive nebula)**

Motion around a fixed axis is the most likely explanation. Why not motionless?

During a gravitational collapse, an angular momentum always remains and its conservation involves an increase of angular velocity. Disc shapes are often to be found in the Universe (galaxies, solar system, ...). The same thing happens when a piece of pizza paste is rotating: viscosity forces give flatness.

Such a model was imagined very early (KANT, LAPLACE) but the weakness of the angular velocity of the Sun round its own axis (1 revolution in 25 days) can't be explained: the Sun contains 99 % of the mass of the solar system but less than 1 % of angular momentum. LAPLACE, who perfectly understands the notion of angular momentum, believes in such a model: the nearly elliptic trajectories of the planets are in the same plane, within a few degrees: the initial nebula hypothesis seems the most probable to him. We have to wait as late as 1960 when Evry SCHATZMANN imagined a loss of angular momentum through a braking due to a coupling between the solar wind and the magnetic field.

Besides, the astronomers detected stars like β Pictoris, Vega, Formalhaut, ε Eridani, ... surrounded with a dust cloud similar to LAPLACE's nebula. That considerably increased astrophysicists' confidence in that model. β Pictoris was studied with great attention: the dust cloud disc was made from the accretion disc which created the star and is spread here and there, on 400 AU from the centre the sky. Dust specks are very small and the cloud is very much like the idea we have about the solar system before planets were born (the solar system is roughly 80 AU wide).

But the birth of planets from a rotating dust disc is a difficult physics problem. Two theories were imagined: gravitational condensation and accretion.

The former (KUIPER, CAMERON) in which planets were supposedly born from instabilities and gravitational collapse has now been dropped for the following reasons:

- It needs a much bigger mass than the solar system's.

- Orders of magnitude for the durations of the different phases.
- Weakness of the hydrogen concentration in small planets.
- A collapse model badly explains the fact that rotation axes of planets on themselves are pointed in any direction.

A few ideas of the model may survive, for instance for giant planets (Jupiter, Saturn) whose core could reach the critical size for gravitational collapse. It's true Jupiter and Saturn are small solar system with as many satellites as solar planets. Neptune et Uranus are not big enough.

3° The theory of collisional accretion (Viktor SAFRONOV, George WETHERHILL and Larry COX)

The latter, planets formation by accretion of planetesimals, is more like LAPLACE's original idea. First works by SCHMIDT are as early as 1940 but it was defended by the Russian school of GUREVITCH, LEVEDINSKII (1950) and especially SAFRONOV (1960) and the American school of GREENBERG, WETHERHILL and COX (1970).

The basic simplifying idea is LAPLACE's: first after the explosion of the supernova, the elements have a uniform circular motion; gravitational interactions make the accretion possible and slightly change the motions which remain near the initial circular motion. Of course the problem isn't the same with microscopic grains and with planets embryos.

- Will scattered matter agglomerate or, on the contrary, disperse even more? Will we get a ring structure or planets? We still only have a glimpse of a solution thanks to the genius of Russian and American theoreticians, the latter being helped by the biggest computers. But even those can only give approximate solutions for very rough models.

Besides, initial conditions are largely unknown and we get either amazing or encouraging results. Nevertheless, it seems reasonable that we can foresee the birth of about ten planets. Larry COX, with about a hundred initial planetesimals of 1.2×10^{25} kg (total mass is internal planets': Mercury to Mars), succeeds, with initial random orbits parameters, in getting 22 embryos within 30 Million years. In 79 Million years, 11 bodies; in 151 Million years 6; in 441 Millions years, only 4 planets (the external planet is the biggest and is the result of the accretion of 54 planetesimals): the formation of the Earth would last a hundred Million years.

Besides, the ring structure can be obtained; but it is unstable, may be stabilised by small pinching satellites (arcs of Neptune rings).

A collision must not be too weak: the accretion would not happen because the relative velocity has to be sufficient to generate the fusion of rocks in contact. But if it is too strong, the planets break.

- What is the relative velocity between two bodies to make the collision efficient? It's **SAFRONOV's hypothesis** which we can justify with homogeneity and pertinence arguments. In a collision, from a little before to a little afterwards, the simplifying hypothesis is to disregard the Sun's and other planets' action compared to the interaction of the bodies which are in collision. For these two projectiles, liberation velocity u_L depends on both kinematics and gravitation; and gravitational interaction between the two bodies will change the paths and make planetesimals collide.

For two bodies of same m mass in interaction: u_L gives a zero energy which just allows one body to go very far with a very small velocity: $\frac{1}{2}mu_L^2 - \frac{Gm^2}{r} = 0$ or, after simplifying, $u_L = \sqrt{\frac{2Gm}{r}}$. For a homogeneous spherical body of volumic mass (density) ρ, $m = \frac{4}{3}\pi\rho r^3$, which gives: $u_L = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{8\pi}{3}G\rho} \times r = 2.4\times10^{5} r\sqrt{\rho} = 1.5\times10^{5} r$ if $\rho = 4000$ kg.m^{-3}.

This liberation velocity increases proportionally to the radius of the attracting body. Therefore, once the accretion has begun, if SAFRONOV's hypothesis is correct, relative velocities of the projectiles have to increase with their sizes. In fact, gravitational attraction perturbs the orbits and tends to increase relative velocities while collisions tend to thermalize the system i.e. to make paths more circular and velocities nearer the mean. It has been verified that the former phenomenon is the stronger: there is a sort of focusing which increase the cross sections as well as relative velocities.

Let us choose reasonable conditions before the collision of two bodies with accretion (relative velocity near liberation velocity, angle between the two absolute velocities of a few degrees). After simplifying the hypothesis, a raising of temperature by a few thousand kelvins makes the fusion of rocks possible, the result depending on the sizes of the interacting bodies.

Calculation can be made for an Earth-Moon collision, since it happened !
LASKAR did point out there is room for a Moon satellite but not for a Moon solar planet.
This lecture ends with that apocalypse. I thank you very much for your attention.

J-F LE BOURHIS
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